COMPUTATIONAL MODELS FOR COMBINED HEAT TRANSFER IN SNOW AND ICE

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The transport radiative transfer equation (RTE) for ~~in~~ a flat layer of isotropic snow can be written as follows (Dombrovsky and Baillis 2010, Dombrovsky 2019):

$$\vec{\Omega}\nabla I_\lambda(\vec{r},\vec{\Omega}) + \beta_\lambda(\vec{r}) I_\lambda(\vec{r},\vec{\Omega}) = \frac{\sigma_\lambda(\vec{r})}{4\pi}\int_{(4\pi)} I_\lambda(\vec{r},\vec{\Omega}')\, \Phi_\lambda(\vec{r},\vec{\Omega}'\cdot\vec{\Omega})\, \mathrm{d}\vec{\Omega}' \tag{1}$$

where I_λ is the spectral radiation intensity at point $\vec{r}$ in direction $\vec{\Omega}$, $\beta_\lambda = \alpha_\lambda + \sigma_\lambda$ is the extinction coefficient, α_λ and σ_λ are the coefficients of absorption and scattering, respectively. The physical meaning of Eq. (1) is evident: variation of the radiation intensity takes place due to extinction and the scattering from other directions.

The thermal radiation of snow and ice should also be taken into account, but only in the infrared part of the spectrum, outside the near-infrared range. Both snow and ice are opaque in the middle-infrared and thermal radiation is emitted only by the surface of the snow cover or ice sheet. Therefore, the intrinsic thermal radiation of the medium does not appear in the radiative transfer equation.

The multiple scattering of solar radiation takes place in snow and scattering ice, which contains numerous small gas bubbles. This makes it possible to use the so-called transport approximation for the single-scattering phase function (Dombrovsky 2012). In this case, it is sufficient to know the asymmetry factor of scattering (the average cosine of the scattering angle), which is independent of polarization:

$$\bar{\mu}_\lambda(\vec{r}) = \frac{1}{4\pi}\int_{(4\pi)} (\vec{\Omega}'\cdot\vec{\Omega})\, \Phi_\lambda(\vec{r},\vec{\Omega}'\cdot\vec{\Omega})\, \mathrm{d}\vec{\Omega}' \tag{2}$$

The transport RTE can be written as:

$$\vec{\Omega}\nabla I_\lambda(\vec{r},\vec{\Omega}) + \beta_\lambda^{\mathrm{tr}} I_\lambda(\vec{r},\vec{\Omega}) = \frac{\sigma_\lambda^{\mathrm{tr}}}{4\pi} G_\lambda(\vec{r}) \qquad G_\lambda(\vec{r}) = \int_{(4\pi)} I_\lambda(\vec{r},\vec{\Omega})\, \mathrm{d}\vec{\Omega} \tag{3}$$

where G_λ is the irradiation function and the transport scattering and extinction coefficients are introduced as follows:

$$\sigma_\lambda^{\mathrm{tr}} = \sigma_\lambda(1-\bar{\mu}_\lambda) \qquad \beta_\lambda^{\mathrm{tr}} = \alpha_\lambda + \sigma_\lambda^{\mathrm{tr}} = \beta_\lambda - \sigma_\lambda\bar{\mu}_\lambda \tag{4}$$

An accurate numerical solution to the RTE in scattering media is a very complicated task even in the case of transport approximation for single-scattering phase function. One can find a number of studies in the literature on specific numerical methods developed to obtain more accurate spatial and angular characteristics of the radiation intensity field. Several modifications of the discrete ordinates method, the finite-volume method, and statistical Monte Carlo (MC) methods are the most popular tools employed by many authors. However, simple and physically clear differential approximations are widely used for solving the radiative transfer problems in scattering media, particularly in combined heat transfer problems (Dombrovsky 2019).

The angular dependence of the radiation intensity is the main difficulty in solving the RTE. Fortunately, this angular dependence is rather simple in many applied problems. This enables one to derive the so-called differential approximations. These approximations have a long history. The latter is reflected in their well-known names such as the Eddington method, the Schwarzschild–Schuster method, etc. It should be emphasized that all the differential approximations are based on one or another simple presentation of angular dependence of the radiation intensity. As a result, it is sufficient to consider a limited number of coordinate functions and turn to the coupled ordinary differential equations by the use of integration of RTE.

The simplest differential approximations, brought together by the general term "diffusion approximation", give the following representation of the spectral radiative flux:

$$\vec{q}_\lambda = -D_\lambda \nabla G_\lambda \tag{5}$$

and differ only by the expression for the radiation diffusion coefficient D_λ. It is known that Eq. (5) can be also derived based on some assumptions concerning the angular dependence of radiation intensity. Substituting Eq. (5) into the radiation energy balance

$$\nabla \vec{q}_\lambda = -\alpha_\lambda G_\lambda \tag{6}$$

we obtain the modified Helmholtz equation for the spectral irradiation:

$$-\nabla(D_\lambda \nabla G_\lambda) + \alpha_\lambda G_\lambda = 0 \tag{7}$$

Note that the radiation power absorbed in a medium is expressed as follows:

$$P(\vec{r}) = \int p_\lambda(\vec{r})\ \mathrm{d}\lambda \qquad\qquad p_\lambda = \alpha_\lambda G_\lambda \tag{8}$$

To clarify the physical sense of the diffusion approximation, consider the linear angular dependence of radiation intensity in the popular P_1 approximation:

$$I_\lambda(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\left[G_\lambda(\vec{r}) + 3\vec{\Omega}\vec{q}_\lambda(\vec{r})\right] \tag{9}$$

By multiplying the transfer equation (3) by $\vec{\Omega}$ and integrating it over a solid angle, one can find that the spectral radiative flux is simply related to the spectral irradiation:

$$\vec{q}_\lambda = -D_\lambda \nabla G_\lambda \qquad\qquad D_\lambda = 1/(3\beta_\lambda^{\mathrm{tr}}) \tag{10}$$

Note that Eq. (10) is obtained for an arbitrary scattering function but it is the same as that for transport approximation.

The two-flux approximation is also often used to solve 1-D radiative transfer problems including those considered in the present paper. Consider the transport RTE:

$$\mu \frac{\partial I_\lambda}{\partial z} + \beta_\lambda^{\mathrm{tr}} I_\lambda = \frac{\sigma_\lambda^{\mathrm{tr}}}{2} G_\lambda \qquad\qquad \mu = cos\,\theta \tag{11}$$

where $0 < z < d$ is the coordinate across the medium layer, d is the layer thickness, and θ is the angle measured from the z-axis direction. The spectral radiative flux is obtained as

$$q_\lambda = (G_\lambda^+ - G_\lambda^-)/2 \qquad\qquad G_\lambda^\pm(\vec{r}) = \pm 2\pi \int_0^{\pm 1} I_\lambda(z,\mu)\, \mathrm{d}\mu \tag{12}$$

Note that $G_\lambda = G_\lambda^- + G_\lambda^+$. With the use of two-flux approximation, the radiation intensity is assumed to be constant over the backward and forward hemispheres:

$$I_\lambda(z,\mu) = \begin{cases} \frac{1}{2\pi} G_\lambda^-(z) \ \text{ when } \ -1 < \mu < 0 \\ \frac{1}{2\pi} G_\lambda^+(z) \ \text{ when } \ 0 < \mu < 1 \end{cases} \tag{13}$$

The integration of RTE over the hemispheres leads to the following coupled equations:

$$-\frac{1}{2}\frac{dG_\lambda^-}{dz} + \beta_\lambda^{\mathrm{tr}} G_\lambda^- = \frac{\sigma_\lambda^{\mathrm{tr}}}{2} G_\lambda \qquad\qquad \frac{1}{2}\frac{dG_\lambda^+}{dz} + \beta_\lambda^{\mathrm{tr}} G_\lambda^+ = \frac{\sigma_\lambda^{\mathrm{tr}}}{2} G_\lambda \tag{14}$$

These equations and the expression for the spectral radiative flux can be written as:

$$-\frac{d}{dz}\left(D_\lambda \frac{dG_\lambda}{dz}\right) + \alpha_\lambda G_\lambda = 0 \qquad\qquad q_\lambda = -D_\lambda \frac{dG_\lambda}{dz} \tag{15}$$

where $D_\lambda = 1/(4\beta_\lambda^{\mathrm{tr}})$. One can see that two-flux approximation leads to the typical relations of the diffusion approximation. We will not discuss the boundary conditions for the P_1 and two-flux equations. The details of these approaches including the accuracy analysis based on a comparison with reference solutions for the model problems can be found in monograph by Dombrovsky and Baillis (2010). However, it is important to recall that P_1 is insufficiently accurate at the medium boundaries in the case of discontinuous angular dependence of radiation intensity. Therefore, the two-flux approximation is preferable when the solar light irradiates a snowpack or ice sheet.

Consider first the radiative transfer in ~~a~~ snow. This problem is relatively simple because there is no any refracting host medium. The solution considered in this section is applicable to the snowpack irradiated by collimated solar radiation and also by diffuse radiation from the sky. In the case of oblique solar irradiation, the radiation field in a snow layer is three-dimensional. However, the distribution of the absorbed power over the thickness of a medium layer depends only on the solar zenith angle and it does not matter which side of the normal the Sun is located. Let us imagine that the radiative flux is uniformly distributed over the surface of a cone with the same zenith angle at the vertex. Of course, in this case the power profile of the absorbed radiation will be the same. In other words, the original problem is equivalent to the 1-D axisymmetric problem of the oblique irradiation along the cone surface. This idea has been used by Dombrovsky et al. (2019). The axial symmetry of the equivalent problem enables one to integrate the original transport RTE over the azimuth angle. The resulting equations can be written as follows:

$$\mu \frac{\partial I_{\lambda,\mathrm{i}}}{\partial z} + \beta_\lambda^{\mathrm{tr}} I_{\lambda,\mathrm{i}} = \frac{\sigma_\lambda^{\mathrm{tr}}}{2} G_{\lambda,\mathrm{i}} \qquad G_{\lambda,\mathrm{i}}(z) = \int_{-1}^{1} I_{\lambda,\mathrm{i}}(z,\mu)\, \mathrm{d}\mu \qquad \mu = \cos\theta \quad 0 < z < d \tag{16a}$$

$$I_{\lambda,\mathrm{i}}(0,\mu) = I_\lambda^{\mathrm{sol}}(\mu_\mathrm{i})\, \delta(\mu_\mathrm{i} - \mu) + I_\lambda^{\mathrm{sky}} \qquad I_{\lambda,\mathrm{i}}(d,-\mu) = 0 \qquad \mu, \mu_\mathrm{i} > 0 \tag{16b}$$

where $\mu_\mathrm{i} = cos\,\theta_\mathrm{i}$ corresponds to the direction of incident collimated light, I_λ^sol is the intensity of solar radiation transmitted through the atmosphere, I_λ^sky is the intensity of diffuse radiation from the sky, d is the thickness of a medium layer. The conditions of solar irradiation depend on current time, t, but the value of t is just a parameter of the radiative transfer problem.

The linearity of the RTE makes possible a separate consideration of additive contribution of the direct solar radiation and radiation from the sky. Consider first the transfer of direct solar radiation using the dimensionless form of Eqs. (16a) and (16b):

$$\mu \frac{\partial \bar{I}_{\lambda,\mathrm{i}}}{\partial \tau_\lambda^\mathrm{tr}} + \bar{I}_{\lambda,\mathrm{i}} = \frac{\omega_\lambda^\mathrm{tr}}{2} \bar{G}_{\lambda,\mathrm{i}} \qquad \bar{G}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr}) = \int_{-1}^{1} \bar{I}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr},\mu)\,\mathrm{d}\mu \tag{17a}$$

$$\bar{I}_{\lambda,\mathrm{i}}(0,\mu) = \delta(\mu_\mathrm{i} - \mu) \qquad \bar{I}_{\lambda,\mathrm{i}}(\infty,-\mu) = 0 \qquad \mu,\mu_\mathrm{i} > 0 \tag{17b}$$

There are some simplifications in these equations: the boundary condition at $z = d$ is replaced by the same condition at $\tau_\lambda^\mathrm{tr} \to \infty$ (this can be done for optically thick layers) and the following dimensionless variables are introduced: the ratio of $\bar{I}_{\lambda,\mathrm{i}} = I_\lambda / I_\lambda^\mathrm{sol}(\mu_\mathrm{i})$, the current optical thickness $\tau_\lambda^\mathrm{tr} = \int_0^z \beta_\lambda^\mathrm{tr}(z)\,\mathrm{d}z$, and transport albedo of single scattering $\omega_\lambda^\mathrm{tr}(z) = \sigma_\lambda^\mathrm{tr}(z)/\beta_\lambda^\mathrm{tr}(z)$.

Following the usual technique, the radiation intensity is represented as a sum of the diffuse component $\bar{J}_{\lambda,\mathrm{i}}$ and the term corresponding to the direct solar radiation:

$$\bar{I}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr},\mu) = \bar{J}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr},\mu) + E_\lambda^i \delta(\mu - \mu_\mathrm{i}) \qquad E_{\lambda,\mathrm{i}} = exp(-\tau_\lambda^\mathrm{tr}/\mu_\mathrm{i}) \tag{18}$$

The spectral irradiance can be also written as a sum of two components:

$$\bar{G}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr}) = \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff}(\tau_\lambda^\mathrm{tr}) + E_{\lambda,\mathrm{i}} \qquad \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} = \int_{-1}^{1} \bar{J}_\lambda\,\mathrm{d}\mu \tag{19}$$

The mathematical problem statement for the diffuse component of radiation intensity is as follows:

$$\mu \frac{\partial \bar{J}_{\lambda,\mathrm{i}}}{\partial \tau_\lambda^\mathrm{tr}} + \bar{J}_{\lambda,\mathrm{i}} = \frac{\omega_\lambda^\mathrm{tr}}{2} \left(\bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} + E_{\lambda,\mathrm{i}} \right) \tag{20a}$$

$$\bar{J}_{\lambda,\mathrm{i}}(0,\mu) = \bar{J}_{\lambda,\mathrm{i}}(\infty,-\mu) = 0 \qquad \mu > 0 \tag{20b}$$

The source term on the right of Eq. (20a) does not depend on angular variable μ. This enables further simplification of the problem with the use of two-flux approximation.

For the diffuse radiation from the sky, one should use another normalization, $\bar{I}_\lambda = I_\lambda / q_\lambda^\mathrm{sky}$, and replace the term $\delta(\mu_\mathrm{i} - \mu)$ by 1 in Eq. (17b). As a result, the following equations are true:

$$\mu \frac{\partial \bar{J}_\lambda}{\partial \tau_\lambda^\mathrm{tr}} + \bar{J}_\lambda = \frac{\omega_\lambda^\mathrm{tr}}{2} \bar{G}_\lambda^\mathrm{sky} \tag{21a}$$

$$\bar{J}_\lambda(0,\mu) = 1 \qquad \bar{J}_\lambda(\infty,-\mu) = 0 \qquad \mu > 0 \tag{21b}$$

The spectral component of the absorbed radiation power taking into account both the direct solar radiation and the diffuse radiation of the sky, can be written as follows:

$$p_\lambda(z) = \alpha_\lambda(z) \left[\bar{G}_{\lambda,\mathrm{i}}(z)\, q_{\lambda,\mathrm{i}}^\mathrm{sol} + \bar{G}_\lambda^\mathrm{sky}(z)\, q_\lambda^\mathrm{sky} \right] \tag{22}$$

According to the two-flux method, the following approximation of the diffuse component of radiation intensity is considered:

$$\bar{J}_{\lambda,\mathrm{i}}(\tau_\lambda^{\mathrm{tr}},\mu) = \begin{cases} \bar{J}_{\lambda,\mathrm{i}}^{-}(\tau_\lambda^{\mathrm{tr}}) & \text{when } -1<\mu<0 \\ \bar{J}_{\lambda,\mathrm{i}}^{+}(\tau_\lambda^{\mathrm{tr}}) & \text{when } 0<\mu<1 \end{cases} \tag{23}$$

Integrating Eq. (20a) separately over the intervals $-1<\mu<0$ and $0<\mu<1$, one can obtain the boundary-value problem for the diffuse irradiance $\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}} = \bar{J}_{\lambda,\mathrm{i}}^{-} + \bar{J}_{\lambda,\mathrm{i}}^{+}$:

$$-\left(\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}}\right)'' + \xi_\lambda^2 \bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}} = 4\omega_\lambda^{\mathrm{tr}} E_{\lambda,\mathrm{i}} \qquad \xi_\lambda = 2\sqrt{1-\omega_\lambda^{\mathrm{tr}}} \tag{24a}$$

$$\left(\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}}\right)'(0) = 2\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}}(0) \qquad \left(\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}}\right)'(\infty) = 0 \tag{24b}$$

The above equations are true at arbitrary variations of ξ_λ and $\omega_\lambda^{\mathrm{tr}}$ with the current optical thickness $\tau_\lambda^{\mathrm{tr}}$, and the problem (24a,b) can be easily solved numerically. However, it is interesting to obtain an analytical solution of the problem in the case of uniform optical properties of snow. The analytical solution to Eqs. (24a,b) at $\xi_\lambda \neq 1/\mu_\mathrm{i}$ can be written as follows (Dombrovsky and Randrianalisoa 2018, Dombrovsky et al. 2019):

$$\bar{G}_{\lambda,\mathrm{i}}^{\mathrm{diff}} = \frac{4\omega_\lambda^{\mathrm{tr}}}{\xi_\lambda^2 - 1/\mu_i^2}\left(E_{\lambda,\mathrm{i}} - \frac{2+1/\mu_i}{2+\xi_\lambda} E_\lambda^{\mathrm{diff}}\right) \qquad E_\lambda^{\mathrm{diff}} = exp(-\xi_\lambda \tau_\lambda^{\mathrm{tr}}) \tag{25}$$

where μ_i should be considered as a parameter. There are two different exponential functions in Eq. (25). The first one, $E_{\lambda,\mathrm{i}}$, is related with a contribution of the collimated solar radiation, whereas the second one, $E_\lambda^{\mathrm{diff}}$, corresponds to the diffuse component of the radiation field.

The following simple expression can be easily obtained for the diffuse radiation of the sky:

$$\bar{G}_\lambda^{\mathrm{sky}}(\tau_\lambda^{\mathrm{tr}}) = 2E_\lambda^{\mathrm{diff}} \tag{26}$$

A comparison of functions $E_{\lambda,\mathrm{i}}(\tau_\lambda^{\mathrm{tr}})$ and $E_\lambda^{\mathrm{diff}}(\tau_\lambda^{\mathrm{tr}})$ indicates that the propagation depth of collimated radiation (defined as a distance from the snow surface at which the irradiance decreases e times) is less than the propagation depth of diffuse radiation component when $\omega_\lambda^{\mathrm{tr}} > 0.75$.

In the case of a semi-transparent scattering ice sheet, the above radiative transfer model should be modified to take into account the refraction of solar light in ice. The generalized boundary condition leads to the following equations for the direct solar radiation instead of Eqs. (16a,b):

$$\mu \frac{\partial \bar{I}_{\lambda,\mathrm{i}}}{\partial \tau_\lambda^{\mathrm{tr}}} + \bar{I}_{\lambda,\mathrm{i}} = \frac{\omega_\lambda^{\mathrm{tr}}}{2}\bar{G}_{\lambda,\mathrm{i}} \qquad \bar{G}_{\lambda,\mathrm{i}}(\tau_\lambda^{\mathrm{tr}}) = \int_{-1}^{1} \bar{I}_{\lambda,\mathrm{i}}(\tau_\lambda^{\mathrm{tr}},\mu)\,\mathrm{d}\mu \tag{27a}$$

$$\bar{I}_{\lambda,\mathrm{i}}(0,\mu) = r_{\lambda,\mathrm{i}}\bar{I}_{\lambda,\mathrm{i}}(0,-\mu) + \left(1-r_{\lambda,\mathrm{i}}\right)\delta\left(\mu_\mathrm{j}-\mu\right) \qquad \bar{I}_{\lambda,\mathrm{i}}(\infty,\mu) = \bar{I}_{\lambda,\mathrm{i}}(\infty,-\mu) = 0 \quad \mu,\mu_\mathrm{j} > 0 \tag{27b}$$

where $\bar{I}_{\lambda,\mathrm{i}} = I_{\lambda,\mathrm{i}}/I_{\lambda,\mathrm{i}}^{\mathrm{inc}}$, $\bar{G}_{\lambda,\mathrm{i}} = G_{\lambda,\mathrm{i}}/I_{\lambda,\mathrm{i}}^{\mathrm{inc}}$, $I_{\lambda,\mathrm{i}}^{\mathrm{inc}}$ is the incident radiation intensity in direction μ_i, $r_{\lambda,\mathrm{i}} = r_\lambda(n,\mu_\mathrm{i})$ is the Fresnel reflection coefficient (Born and Wolf 1999), μ_j is the directional cosine of the angle of refraction. The refraction leads to the change of direction of the incident solar radiation at the medium boundary. The new direction of the refracted solar light under the horizontal surface of a refracting medium is $\mu_\mathrm{j} = \sqrt{1-(1-\mu_\mathrm{i}^2)/n^2}$. In particular, sunlight at sunrise and sunset propagates in a horizontal layer of refracting and non-scattering medium along the surface of a cone with a half-angle at the apex $\theta = arccos(1 - 1/n^2)$. Note that simple equation of $r_{\lambda,\mathrm{i}} = r_{\lambda,\mathrm{n}} = (n-1)^2/(n+1)^2$ can be used instead of the general Fresnel equations in the case of normal irradiation ($\mu_\mathrm{i} = 1$) of a weakly absorbing host medium, when $\kappa << n$.

As before, the radiation intensity is presented as a sum of the diffuse and collimated components:

$$\bar{I}_{\lambda,\mathrm{i}} = \bar{J}_{\lambda,\mathrm{i}} + \left(1 - r_{\lambda,\mathrm{i}}\right) E_{\lambda,\mathrm{j}} \delta\left(\mu_\mathrm{j} - \mu\right) \quad E_{\lambda,\mathrm{j}} = exp\left(-\tau_\lambda^\mathrm{tr}/\mu_\mathrm{j}\right) \tag{28}$$

and the problem statement for the diffuse component is as follows:

$$\mu \frac{\partial \bar{J}_{\lambda,\mathrm{i}}}{\partial \tau_\lambda^\mathrm{tr}} + \bar{J}_{\lambda,\mathrm{i}} = \frac{\omega_\lambda^\mathrm{tr}}{2} \bar{G}_{\lambda,\mathrm{i}} \qquad \bar{G}_{\lambda,\mathrm{i}} = \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} + \left(1 - r_{\lambda,\mathrm{i}}\right) E_{\lambda,\mathrm{j}} \tag{29a}$$

$$\bar{J}_{\lambda,\mathrm{i}}(0,\mu) = r_{\lambda,\mathrm{i}} \bar{J}_{\lambda,\mathrm{i}}(0,-\mu) \qquad \bar{J}_{\lambda,\mathrm{i}}(\infty,-\mu) = \bar{J}_{\lambda,\mathrm{i}}(\infty,-\mu) \qquad \mu, \mu_\mathrm{i} > 0 \tag{29b}$$

According to the modified two-flux approximation (Dombrovsky et al. 2006) the angular dependence of the radiation intensity takes into account the total internal reflection at $\tau_\lambda^\mathrm{tr} = 0$:

$$\bar{J}_{\lambda,\mathrm{i}}(\tau_\lambda^\mathrm{tr}, \mu) = \begin{cases} \bar{J}_{\lambda,\mathrm{i}}^{-}(\tau_\lambda^\mathrm{tr}) \ \text{when} \ -1 \le \mu < -\mu_\mathrm{c} \\ \bar{J}_{\lambda,\mathrm{i}}^{0}(\tau_\lambda^\mathrm{tr}) \ \text{when} \ -\mu_\mathrm{c} < \mu < \mu_\mathrm{c} \\ \bar{J}_{\lambda,\mathrm{i}}^{+}(\tau_\lambda^\mathrm{tr}) \ \text{when} \ \mu_\mathrm{c} < \mu \le 1 \end{cases} \qquad \mu_\mathrm{c} = \sqrt{1 - \frac{1}{n^2}} \tag{30}$$

The term "two-flux" is applicable to this approximation because the interval of $-\mu_\mathrm{c} < \mu < \mu_\mathrm{c}$ does not contribute to the radiative flux. Integrating Eq. (27a) over the angles, one can obtain the boundary-value problem for the normalized diffuse irradiance:

$$-\left(\bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff}\right)'' + \xi_\lambda^2 \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} = \xi_\lambda^2 \chi_i E_{\lambda,\mathrm{j}} \tag{31a}$$

$$\tau_\lambda^\mathrm{tr} = 0, \quad (1 + \mu_\mathrm{c})\left(\bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff}\right)' = 2\gamma \, \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} \qquad \tau_\lambda^\mathrm{tr} = \infty, \quad \bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} = 0 \tag{31b}$$

$$\xi_\lambda^2 = \frac{4}{(1+\mu_\mathrm{c})^2} \frac{1-\omega_\lambda^\mathrm{tr}}{1-\omega_\lambda^\mathrm{tr}\mu_\mathrm{c}} \quad \chi_\mathrm{i} = \frac{\omega_\lambda^\mathrm{tr}}{1-\omega_\lambda^\mathrm{tr}}\left(1 - r_{\lambda,\mathrm{i}}\right) \quad \gamma = \frac{1-\bar{r}_\lambda}{1+\bar{r}_\lambda} \tag{31c}$$

where $\bar{r}_\lambda$ is the angle-averaged reflectance for the diffuse radiation at the interface $\tau_\lambda^\mathrm{tr} = 0$.
The analytical solution to the above problem at $\xi_\lambda \neq 1/\mu_\mathrm{j}$ is:

$$\bar{G}_{\lambda,\mathrm{i}}^\mathrm{diff} = \frac{\xi_\lambda^2 \chi_\mathrm{i}}{\xi_\lambda^2 - 1/\mu_\mathrm{j}^2}\left(E_{\lambda,\mathrm{j}} - \frac{2\gamma + (1+\mu_\mathrm{c})/\mu_\mathrm{j}}{2\gamma + (1+\mu_\mathrm{c})\xi_\lambda} E_{\lambda,\mathrm{i}}^\mathrm{diff}\right) \tag{32}$$

In the case of a non-refracting medium ($r_{\lambda,\mathrm{i}} = 0$, $\mu_\mathrm{j} = \mu_\mathrm{i}$, and $\gamma = 1$), Eqs. (31a-c) and (32) were used by Dombrovsky et al. (2019) and Dombrovsky and Kokhanovsky (2019) for solar radiation transfer in a snowpack.

It should be recalled that a comparison with the exact numerical results obtained using the high-order composite discrete ordinates method (CDOM) performed by Dombrovsky et al. (2006) and also a comparison with the Monte Carlo simulation showed that the modified two-flux approximation is sufficiently accurate for the use in diverse applications (Dombrovsky, 2019).

The ordinary or modified two-flux approximation is usually a good approach for estimating the irradiation field. With the use of transport approximation, the irradiation field is sufficient to obtain the source function (the right-hand side) of RTE. As a result, there is a possibility to improve the solution using the second iterative step of a computational procedure. It is sufficient to solve the RTE with the known source function using one of the ray-tracing procedures. A study of applicability and accuracy of the combined two-step procedure with the use of Monte Carlo method at the second step of the iterative solution has been reported in the book by Dombrovsky and Baillis (2010).

Consider now the 1-D heat transfer problem. Generally speaking, there are various thermal or related processes in a snowpack or scattering ice sheet and many of these processes should be involved in a complete transient computational model for snow heating. As an example, one should recall the processes of ice sublimation and diffusion of water vapor through a snow layer. These processes appear to be important for the snow microstructure, which determines macroscopic snow properties. However, this is beyond the scope of the present article.

The transient energy equation for temperature, $T(t,z)$, in a layer of snow or ice and the accompanying initial and boundary conditions can be written as follows:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + P \qquad t > 0 \quad 0 < z < d_{\mathrm{th}} \tag{33a}$$

$$T(0,z) = T_0(z) \tag{33b}$$

$$z = 0, \quad -k\frac{\partial T}{\partial z} = h\,(T_{\mathrm{air}} - T) - \varepsilon\,\pi \int_{\lambda_{\mathrm{w1}}}^{\lambda_{\mathrm{w2}}} I_{\mathrm{b}}(T,\lambda)\,\mathrm{d}\lambda \qquad z = d_{\mathrm{th}}, \quad \frac{\partial T}{\partial z} = 0 \tag{33c}$$

where z is the coordinate measured from the irradiated surface of the layer, ρ, c, and k are the density, the specific (per unit mass) heat capacity, and the thermal conductivity of snow or ice, $T_{\mathrm{air}}(t)$ is the temperature of ambient air, and $h(t)$ is the convective heat transfer coefficient. The adiabatic condition at the boundary $z = d_{\mathrm{th}} >> d$ means that we neglect heat transfer at $z > d_{\mathrm{th}}$. Of course, the value of d_{th} increases with time and should be estimated using additional calculations. The time dependence of heat transfer coefficient is determined by a wind speed.

The last term in the right-hand side of the boundary condition at $z = 0$ is the mid-infrared radiative cooling due to thermal radiation of snowpack or ice sheet surface in the atmospheric window of $\lambda_{\mathrm{w1}} < \lambda < \lambda_{\mathrm{w2}}$ ($\lambda_{\mathrm{w1}} = 8\mu\mathrm{m}$, $\lambda_{\mathrm{w2}} = 13\mu\mathrm{m}$) (Hossain and Gu 2016). The radiative cooling is limited during the night time when it is not compensated by solar mid-infrared radiation. Therefore, the coefficient ε varies from zero in the day time to the unity at night. The absorbed radiation power, $P(z)$, at arbitrary conditions of solar irradiation should be recalculated time by time during the combined problem solution.

Of course, the great value of the latent heat of ice melting, $L = 0.34\mathrm{MJ/kg}$, should be taken into account. This can be done using an equivalent additional heat capacity, Δc, in narrow temperature range of $T_{\mathrm{m}} - \Delta T < T < T_{\mathrm{m}} + \Delta T$ (where $T_{\mathrm{m}} = 273\mathrm{K}$ is the ice melting temperature, $\Delta T << T_{\mathrm{m}} - T_0$). The following simple variant of this approach was used by Dombrovsky et al. (2019):

$$\Delta c = \frac{L}{\Delta T}\left(1 - |T_{\mathrm{m}} - T|/\Delta T\right) \tag{34}$$

A correct choice of ΔT value is determined by the interval of the computational grid and the time step of numerical solution to problem (33a)–(33c). Obviously, it is difficult to choose a realistic initial profile of temperature, $T_0(z)$, for the heat transfer calculations. Fortunately, the effect of this temperature profile decreases with time. It was shown by the authors that the choice of $T_0(z)$ makes no difference for the snow temperature after about four hours from the conventional initial time moment. An implicit finite-difference scheme of the second order of spatial approximation can be recommended for numerical calculations. The specific combination of a strong decrease of heat generation rate with the distance from the irradiated surface and rather deep propagation of heat in the snow or ice layer due to long-time conduction makes it necessary to use a detailed non-uniform computational grid and double precision calculations.

Note that the above model of heat transfer is incomplete in the case of a considerable melting of snow because water penetrates downward through porous snow and solidifies there. This process should be taken into account in more general physical and computational model. The accompanying variation of optical properties of a non-uniform snowpack should be also analyzed on the basis of more sophisticated models.

It is important that the heat transfer problem considered is not a classical problem of radiative-conductive heat transfer (Dombrovsky and Baillis 2010). In the case of solar heating of a snow cover, the snow temperature does not directly affect the radiative transfer. This simplifies significantly the solution of the problem, since a single spectral calculation of the radiative transfer is sufficient.

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